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A SHORT CONSTRUCTION OF MSM RINGS

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Abstract

Let us term the ring of a strict Morita context as an msm (Morita similar matrix) ring. We can always get an msm ring from an arbitrary Morita context. The aim here is to develop a very short and a direct technique to get such rings.

Let $K(A, B) = [A, B, M, N, \langle\, , \rangle_A, \langle\, , \rangle_B]$ be a Morita context (in short *mc*) in which A and B are associative rings with the multiplicative identities 1_A and 1_B, respectively; M and N are (B, A)- and (A, B)-bimodules, respectively; and $\langle\, , \rangle_A : N \otimes_B M \to A$ and $\langle\, , \rangle_B : M \otimes_A N \to B$ are bimodule morphisms satisfying the associativity conditions:

(i) $m'\langle n, m\rangle_A = \langle m', n\rangle_B m$,

(ii) $\langle n, m\rangle_A n' = n\langle m, n'\rangle_B$,

where $m, m' \in M$ and $n, n' \in N$. If the mc maps $\langle , \rangle_A$ and $\langle , \rangle_B$ are onto, then they become isomorphisms and the *mc* is termed as a *strict Morita context* [1] or a *pmc* (*projective Morita context*) [2] and the matrix ring

$$R = \begin{bmatrix} A & N \\ M & B \end{bmatrix}$$

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is termed as a Morita similar matrix ring (in short an *msm* ring). We present here a very short and a direct technique to construct such rings from any arbitrarily given mc. This is an improved method of the iteration as done in [2, Theorem 2.1].

Let us write $R = R_2$ and set

$$R_3 = \begin{bmatrix} A & N & A \\ M & B & M \\ A & N & A \end{bmatrix},$$

which is a ring under element wise addition and in multiplication we only consider that $NM = \langle N, M\rangle_A \trianglelefteq A$ and $MN = \langle M, N\rangle_B \trianglelefteq B$.

Let us have the datum

$$K(R_2, R_3) = [R_2, M_{32}, N_{23}, R_3, \langle\, , \rangle_{R_2}, \langle\, , \rangle_{R_3}]$$

in which $M_{32} = \begin{bmatrix} A & N \\ M & B \\ A & N \end{bmatrix}$ is an (R_3, R_2)-bimodule; $N_{23} = \begin{bmatrix} A & N & A \\ M & B & M \end{bmatrix}$ is an (R_2, R_3)-bimodule; and the maps $\langle , \rangle_{R_2}$ and $\langle , \rangle_{R_3}$ defined by the formulas

$$\left\langle \begin{bmatrix} a_{11} & n_{12} & a_{13} \\ m_{21} & b_{22} & m_{23} \end{bmatrix}, \begin{bmatrix} a'_{11} & n'_{12} \\ m'_{21} & b'_{22} \\ a'_{31} & n'_{32} \end{bmatrix} \right\rangle_{R_2}$$

$$= \begin{bmatrix} a_{11}a'_{11} + \langle n_{12}, m'_{21}\rangle_A + a_{13}a'_{31} & a_{11}n'_{12} + n_{12}b'_{22} + a_{13}n'_{32} \\ m_{21}a'_{11} + b_{22}m'_{21} + m_{23}a'_{31} & \langle m_{21}, n'_{12}\rangle_B + b_{22}b'_{22} + \langle m_{23}, n'_{32}\rangle_B \end{bmatrix}$$

and

$$\left\langle \begin{bmatrix} a_{11} & n_{12} \\ m_{21} & b_{22} \\ a_{31} & n_{32} \end{bmatrix}, \begin{bmatrix} a'_{11} & n'_{12} & a'_{13} \\ m'_{21} & b'_{22} & m'_{23} \end{bmatrix} \right\rangle_{R_3}$$

$$= \begin{bmatrix} a_{11}a'_{11} + \langle n_{12}, m'_{21}\rangle_A & a_{11}n'_{12} + n_{12}b'_{22} & a_{11}a'_{13} + \langle n_{12}, m'_{23}\rangle_A \\ m_{21}a'_{11} + b_{22}m'_{21} & \langle m_{21}, n'_{12}\rangle_B + b_{22}b'_{22} & m_{21}a'_{13} + b_{22}m'_{23} \\ a_{31}a'_{11} + \langle n_{32}, m'_{21}\rangle_A & a_{31}n'_{12} + n_{32}b'_{22} & a_{31}a'_{13} + \langle n_{32}, m'_{23}\rangle_A \end{bmatrix}$$

are mc maps. One can easily verify the existence of both associativity conditions for above maps and so $K(R_2, R_3)$ is an mc.

Theorem. *Every mc gives rise to an msm ring.*

Proof. One can always get the mc $K(R_2, R_3)$ from any arbitrarily given mc $K(A, B)$. We will demonstrate that $K(R_2, R_3)$ is strict.

In order to prove that $\langle\, , \rangle_{R_2}$ is epic, assume that $\begin{bmatrix} a & n \\ m & b \end{bmatrix} \in R_2$. Set $t \in N_{23} \otimes_{R_3} M_{32}$ in the form

$$t = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & n \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ m & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then clearly,

$$\langle\, , \rangle_{R_2}(t) = \begin{bmatrix} a & n \\ m & b \end{bmatrix}.$$

Similarly, if we set

$$t = \begin{bmatrix} a_{11} & n_{12} \\ m_{21} & b_{22} \\ a_{31} & n_{32} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} a_{13} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{33} & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & m_{23} \end{bmatrix} \in M_{32} \otimes_{R_2} N_{23},$$

then

$$\langle\, , \rangle_{R_3}(t) = \begin{bmatrix} a_{11} & n_{12} & a_{13} \\ m_{21} & b_{22} & m_{23} \\ a_{31} & n_{32} & a_{33} \end{bmatrix},$$

which is an arbitrary element in R_3.

Hence, both mc maps of $K(R_2, R_3)$ are epimorphisms so $K(R_2, R_3)$ is strict. The msm ring obtained from $K(R_2, R_3)$ is

$$R_5 = \begin{bmatrix} R_2 & N_{23} \\ M_{32} & R_3 \end{bmatrix} = \begin{bmatrix} A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \end{bmatrix}.$$

■

Let $K(A, B)$ be an *mc*. From this *mc* we can form the sequence

$$\{A = R_1, R_2, R_3, R_5, \dots, R_{2k+1}, \dots\}.$$

Define a map $\alpha_{j(j+1)} : R_j \to R_{j+1}$ by:

$$\alpha_{j(j+1)}(r_j) = \begin{bmatrix} r_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\alpha_{j(j+1)}$ is a well-defined homomorphism (not identity preserving). In fact, this is a (non-unital) embedding of R_j into R_{j+1}. The transitivity of this process yields the increasing sequence

$$R_1 \le R_2 \le R_3 \le R_4 \le \cdots \le R_n \le \cdots.$$

The upper bound of this ascending sequence is

$$\overline{R} = \bigcup_{i=1}^{\infty} R_i.$$

It is important to note that each matrix in $\overline{R}$ has finite size.

In above sequence it is observed that the ring R_1 in general is not Morita similar to R_2. But as we have proved in Theorem that R_2 is Morita similar to R_3. Thus, R_5 is an *msm ring*. Since the relation *Morita similar* is transitive, it is clear that in the ascending sequence

$$R_5 \le R_7 \le \cdots \le R_{2n+1} \le \cdots,$$

every member is an *msm ring*. Moreover, the context ring of $K(R_2, R_2)$ is R_4, and R_2 is Morita similar to R_2, so R_4 is an *msm ring*. Finally, the Morita ring of $K(\overline{R}, \overline{R})$ is $\overline{R}$, and as $\overline{R}$, is Morita similar to itself, so $\overline{R}$ is an *msm ring*. Hence we conclude that

Corollary. *Let $K(A, B)$ be any mc. Then there is an ascending sequence of the msm rings*

$$R_4 \le R_5 \le \cdots \le R_n \le \cdots$$

including the upper bound $\overline{R}$ which is also an msm ring.

References

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