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# Evolutionary Spectrum and Bispectrum Estimation of a Nonstationary Process

Abdullah I. Al-Shoshan

*College of Computer and Information, Qassim University, Qassim, Saudi Arabia.*
*shoshan@myway.com*

*Abstract.* The modeling of a non-stationary process is a problem of great interest and has many applications in diverse fields in signal processing. In this paper, an algorithm for estimating the shift-variant (SV) parameters of a non-stationary process, using orthogonal projection, is proposed. These parameters can be used for estimating the evolutionary spectrum (ES), a time-frequency distribution, of a non-stationary process. The ES is then estimated with an order based on the dimensionality of the expanding orthogonal polynomials (OPs). In the proposed method the process is decomposed to its basic elements by projecting it, recursively, onto elementary OPs. Simulations are presented when the process has either single or multiple SV parameters. We have also shown that if the output of the SV system is corrupted by stationary/non-stationary noise with symmetric distribution, the noise can be removed using the evolutionary bispectrum (EB), and simulation is given to show the powerful of the EB.

*Keywords*: Non-Stationary Process, Shift-Variant (SV) parameters, Multi-component Signal Analysis, Orthogonal Polynomial (OP), Time-Frequency Distribution (TFD), Evolutionary Spectrum (ES), Evolutionary Bispectrum (EB).

## 1. Introduction

The modeling of a non-stationary process and the estimation of its Time-Frequency Distribution (TFD) are problems of great interest in diverse fields in signal processing and have many applications in speech and image processing, geophysics, bio-statistical signal processing, radar, medicine, and seismology [1-4]. For each application, the modeling results in a set of mathematical equations, which can be used to understand the behavior of the process. When the process is non-stationary, neither the classical power spectrum nor the bi-spectrum can handle the modeling problem since they do not reflect the time variation of the process characteristics [5]. An important issue in the modeling of a nonstationary process is the knowledge of the degree of the process nonstationarity [1-3]. In this paper, we will show that this can be reflected by the expansion order of the SV parameters embedded in the process. Hence, depending on the expansion order, the time-frequency (TF) kernel of the ES can be estimated. This method is proposed for estimating the order of expansion that will be used to give the minimum order needed for estimating the SV, and accordingly the optimum TF kernel, and thus the ES of the energy of the non-stationary process over time. In the proposed method the process is decomposed by its basic elements by projecting it, recursively, onto its elementary OPs.

In previous works, experimental data and functions are represented or approximated by linear combinations of some predefined basis functions, which are usually of polynomial forms, bounded and closed under translation and scaling [5-9]. For these polynomials to exactly be able to represent a given data sequence, they must be able to span the whole space of the data sequence [9-11]. In the expansion approach, a set of OPs such as Hahn, Legendre, Laguerre, Jacobi, or Hermite polynomials can be used. However we need to know or estimate the number of OPs to be used in the fitting problem [12-13], and any time-frequency estimator, like evolutionary periodogram (EP) [14], short-time Fourier transform (STFT) and modified group delay function (MGDF) [16], multi-window Gabor expansion [20], multitaper marginal time–frequency distributions [17], Hartley S-transform [18], or maximum entropy (ME) [19] can be used. Mathematical expressions elucidating the analogy between the wavelets based spectral representation and the traditional one involving trigonometric functions are derived in [15]. In this paper, the Legendre OPs are used and the EP, an estimator of the evolutionary spectrum [8], is applied for estimating the ES of the process. The formal derivation of the ES is explained in Section 1.1 and the derivation of the EB is discussed in Section 1.2.

### *1.1 Evolutionary Spectrum*

According to the Wold-Cramer representation, a non-stationary process, *x(n)*, admits a representation of the form

$$x(n) = \int_{-p}^{p} A(n,w)e^{jwn}dZ(w)$$

where *A(n,w)* is a slowly varying function of time. Define [14]

$$H(n,w) = A(n,w)dZ(w)$$

then the ES of *x(n)* is given by

$$S_x(n,w) = E\{|H(n,w)|^2\}$$

Consider that *x(n)* can be modeled at a frequency of interest $w_0$ as

$$\begin{aligned} x(n) &= x_0(n) + y_{w_0}(n) \\ &= A(n,w_0)e^{jw_0 n}dZ(w_0) + y_{w_0}(n) \qquad (1) \\ &= H(n,w_0)e^{jw_0 n} + y_{w_0}(n) \end{aligned}$$

where $x_0(n)$ is the process component at the desired frequency $w_0$ and $y_{w_0}(n)$ is a zero-mean modeling error including other components of *x(n)* at the frequency of interest. The ES is then obtained for finite length of signals, $\{x(n) | 0 \le n \le N\}$, by assuming that $H(n,w_0)$ can be expressed in time as expansion of OP functions, $\{b_m(n) | 1 \le m \le M, 0 \le n \le N-1\}$ so that

$$H(n,w_0) = \sum_{m=1}^{M} a_m b_m(n) = b(n)a$$

where $a = [a_1, a_2, ..., a_M]^t$ is a vector of random expansion coefficients and $b(n) = [b_1(n), b_2(n), ..., b_M(n)]$ is a vector of the OP functions at time $n$, where $t$ stands for matrix transpose. After manipulating the above equations we get the EP estimator of the ES expressed as [14]

$$\hat{S}_{EP}(n,w) = \frac{N}{M} |\hat{H}(n,w)|^2 \quad 0 \le n \le N-1$$

where $\hat{H}(n,w)$ is an estimate of the TF kernel and can be estimated as follows:

$$\hat{H}(n,w) = \sum_{k=0}^{N-1} w_n(k) x(k) e^{-jwk} \qquad (2)$$

which uses a window that varies with time and constructed from a combination of OPs such that

$$w_n(k) = \sum_{i=1}^{M} b_i^*(n) b_i(k) \qquad (3)$$

The variation of $w_n(k)$ depends on the value of *M*, which is the order of the OP set used. Although the parameter *M* plays an essential role in estimating the EP, no way so far has been proposed for finding the optimum value of *M* [20-22]. Actually, *M* is a process-dependent parameter, which is connected directly to the degree of nonstationarity of the process; therefore, this problem will be investigated throughout this paper. When *x(n)* is stationary, then the optimum value of *M* is one. However, when *x(n)* deviates from stationarity, *M* should be increased accordingly, depending on how far *x(n)* is from stationarity. Some simulation for modeling a non-stationary process is presented in Section 2 when the process has either single or multiple SV parameters.

### *1.2 Evolutionary Bispectrum*

When the additive noise *h(n)* shown in Fig. 1 is non-stationary, the traditional stationary techniques will not be able to remove this type of noise. However it will simply reduce the effect of non-stationary noise with symmetric probability density like Gaussian, Laplace, uniform, and Bernouli-Gaussian. To remove these types of noise, we propose using the evolutionary bispectrum (EB) introduced by Priestley [8].

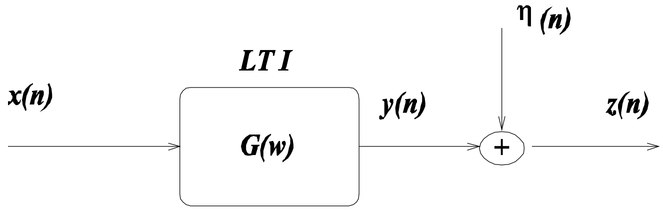



**Fig. 1. A noisy output of an LTI system.**

The EB is defined as follows [8, 9]. The third-order moment of *x(n)* is given by

$$R(n,m_1,m_2) = E\{x(n)x(n+m_1)x(n+m_2)\}$$

which, by using the Wold-Cramer representation, becomes

$$R(n,m_1,m_2) = \int_{-p}^{p}\int_{-p}^{p}\int_{-p}^{p} H_1.P.\ E\{dZ(w_1)dZ(w_2)dZ(w_3)\}$$

where:

$$H_1 = H_x(n,w_1)H_x(n+m_1,w_2)H_x(n+m_2,w_3),$$

$$P = e^{j(w_2m_1+w_3m_2)}e^{jn(w_1+w_2+w_3)}$$

and

$$E\{dZ(w_1)dZ(w_2)dZ(w_3)\} = \begin{cases} S_e(w_1,w_2)dw_1dw_2 & ,w_1+w_2+w_3=0 \\ 0 & ,w_1+w_2+w_3 \neq 0 \end{cases}$$

*i.e.*, it vanishes except along the plane $w_1+w_2+w_3=0$, where $S_e(w_1,w_2)$ is the bispectrum of a stationary white noise, *e*(*n*), then we have that

$$R(n,m_1,m_2) = \int_{-p}^{p}\int_{-p}^{p} H_2.S_e(w_1,w_2)e^{j(w_2m_1-(w_1+w_2)m_2)}\,dw_1dw_2$$

where:

$$H_2 = H_x(n,w_1)H_x(n+m_1,w_2)H_x(n+m_2,-w_1-w_2)$$

and setting $m_1=m_2=0$, we get

$$R(n,0,0) \quad = \quad E\{x(n)^3\} = \int_{-p}^{p}\int_{-p}^{p} S_x(n,w_1,w_2)\,dw_1dw_2 \qquad (4)$$

therefore, the EB of *x(n)* is defined as

$$S_x(n,w_1,w_2) \quad = \quad H_x(n,w_1)H_x(n,w_2)H_x(n,-w_1-w_2)S_e(w_1,w_2) \quad (5)$$

Assuming that *e(n)* is a zero-mean non-Gaussian white noise with $E\{e(n)e(n+m)\,(n+k)\}=\delta(m,k)$, then $S_e(w_1,w_2)=1$ and therefore Eq. (5) becomes

$$S_x(n,w_1,w_2) \quad = \quad H_x(n,w_1)H_x(n,w_2)H_x(n,-w_1-w_2) \quad (6)$$

When *e(n)* is zero-mean Gaussian, then $S_e(w_1,w_2)=0$ and accordingly $S_x(n,w_1,w_2)=0$. Therefore, the EB of a zero-mean non-stationary Gaussian process is identically zero. If the system $h_x(n,m)$ is not time varying, then $H_x(n,w)$ will also be time-invariant, therefore, the EB in this case will reduce to

$$S_x(w_1,w_2) \quad = \quad H_x(w_1)H_x(w_2)H_x(-w_1-w_2) \quad (7)$$

which has a similar interpretation to that of the conventional bispectrum of a stationary process, namely that it is the triple product of the $H_x(n,w)$ at frequencies $w_1$, $w_2$, and $(-w_1-w_2)$. Let *y(n)* be a zero-mean, non-Gaussian process corrupted by an independent, identically distributed (i.i.d.) zero-mean, Gaussian noise, *h(n)*, such that

$$z(n) = y(n) + h(n) \quad (8)$$

where: *y(n)* and *h(n)* are independent, then the third order moment of *z(n)* is

$$E_z = E_y + E_h \quad (9)$$

where: $E\{.\}$ stands for the expected value, $E_z=E\{z(n)z(n+m_1)z(n+m_2)\}$, $E_y=E\{y(n)y(n+m_1)y(n+m_2)\}$, and $E_x=E\{x(n)x(n+m_1)x(n+m_2)\}$. Since *y(n)* has a zero-mean value, its first, second, and third order time-varying cumulants (TVCs) reduce to its first, second, and third order time-varying moments, respectively, so

$$c^y(n,m_1,m_2) = E\{y(n)y(n+m_1)y(n+m_2)\}$$

therefore, the third-order TVC of *z(n)* becomes

$$c^z(n,m_1,m_2) = c^y(n,m_1,m_2) + c^h(n,m_1,m_2) \quad (10)$$

and

$$E\{z(n)^3\} = E\{y(n)^3\} + E\{h(n)^3\} \quad (11)$$

but since *h(n)* is a zero-mean, Gaussian noise, then $E\{h(n)^3\}=0$ and $S_h(n,w_1,w_2)=0$, so

$$E\{z(n)^3\} = E\{y(n)^3\} \quad (12)$$

and hence

$$S_z(n,w_1,w_2) = S_y(n,w_1,w_2) \quad (13)$$

which indicates that the additive Gaussian noise does not affect the third-order TVC or the EB of *y(n)*. We will show later an example that the additive i.i.d. zero-mean non-stationary Gaussian noise can be removed using the EB. In Section 2, the analysis of a process with a single component is presented. Also, a generalized form for estimating the order of nonstationarity process with multiple SV parameters is presented in this section. Some simulations for both cases and for using the EB are given in Section 3. Section 4 concludes the paper.

## 2. Analysis of a Signal with a Single and Multiple SV Parameters

### *2.1 Analysis of a Signal with a Single SV Parameter*

Consider a linear-phase process *x(n)* with only one SV parameter *A(n)* at $w=w_0$ such that

$$x(n) = A(n)e^{jw_0 n} \tag{14}$$

where *A(n)* can be expanded using *M* OPs set $\{b_i(n)\}$ such that

$$A(n) = \sum_{i=1}^{M_0} a_i b_i(n) \tag{15}$$

where the set $\{a_i\}$ are the expansion coefficients of *A(n)*, and $M_0$ is the expansion order. Demodulating *x(n)* by multiplying it by $e^{-j\hat{w}_0 n}$ will lead to

$$\hat{x}_0(n) = A(n)e^{j(w_0-\hat{w}_0)n} \tag{16}$$

where $\hat{w}_0$, if unknown, is estimated from the conventional spectrum of the process and the difference between $w_0$ and $\hat{w}_0$ is close to zero when the spectral estimator is good enough. Assuming that the set of OPs $\{b_i(n)\}$ are known and using their orthogonality property, then, using the inner product associated with them, we get

$$\begin{aligned} D(k) &= \sum_{n=0}^{N-1} b_k(n)x_0(n) = \sum_{n=0}^{N-1} b_k(n)A(n) \\ &= \sum_{i=1}^{M_0} a_i \sum_{n=0}^{N-1} b_k(n)b_i(n) = \begin{cases} a_k & k \le M_0 \\ 0 & k > M_0 \end{cases} \end{aligned} \tag{17}$$

where $a_k$=0 for $k > M_0$ and we have used the property of orthogonality

$$\sum_{n=0}^{N-1} b_k(n)b_i(n) = d(k-i) \tag{18}$$

Therefore, if we increase *k* until we reach *D*(*k*)=0 we will get the value of $M_0$ as the supremum of *k*. Hence if we define a set $K=\{k \mid D(k) \neq 0\}$, the minimum expansion order of the SV parameter *A(n)* is defined as $\hat{M}_0 = sup_k(K)$. After estimating $M_0$, one then estimates the ES of *x(n)* using an expansion order *M* equals to $\hat{M}_0$.

### *2.2 Analysis of a Signal with Multiple SV Parameters*

Assuming that the process under consideration has more than one SV parameter, then using the sinusoidal representation of a SV spectra process, $x(n)$ can be represented as

$$x(n) = \sum_{k=0}^{R-1} A_k(n) e^{jw_k n} \tag{19}$$

where: $R$ is the number of sinusoidals in $x(n)$, and $A_k(n)$, the SV parameter of the $k^{th}$ sinusoidal, is represented as

$$A_k(n) = \sum_{i=1}^{M_k} a_{ki} b_i(n) \quad 0 \le k \le R-1 \tag{20}$$

where: $\{a_{ki}\}$ are the composition coefficients of $A_k(n)$, $\{b_i(n)\}$ is a set of OPs, and $M_k$ is the number of coefficients used for expanding $A_k(n)$. To resolve these SV parameters, we demodulate $x(n)$ by $e^{-jw_r n}$ to get the sub-process

$$x_r(n) = \sum_{k=0}^{R-1} A_k(n) e^{j(w_k - w_r)n} = A_r(n) + \sum_{\substack{k=0 \\ k \ne r}}^{R-1} A_k(n) e^{j(w_k - w_r)n} \tag{21}$$

and then passing $x_r(n)$ through a low pass filter $\{g_r(n)\}$ whose bandwidth is larger than the bandwidth of $A_r(n)$ and assuming a non-overlapping case, we get

$$\tilde{x}_r(n) = A_r(n) = \sum_{i=1}^{M_r} a_{ri} b_i(n) \tag{22}$$

and decomposing $\tilde{x}_r(n)$ using the OP set $\{b_i(n)\}$, as shown in Fig. 2, we get from the inner product

$$\begin{aligned} C_r(k) &= \sum_{n=0}^{N-1} b_k^*(n) \tilde{x}_r(n) = \sum_{n=0}^{N-1} b_k^*(n) A_r(n) \\ &= \sum_{n=0}^{N-1} \sum_{i=1}^{M_r} a_{ri} b_k^*(n) b_i(n) = a_{rk}[u(k-1) - u(k - M_r - 1)] \end{aligned} \tag{23}$$

where: $u(k)$ is the discrete-time unit-step function. Since $a_{rk}=0$ for $k>M_r$, $C_r(k)=0$ for $k>M_r$ and $r>R-1$. Therefore, the decomposition order $k$ should be increased until

$$C_r(k) = C_r(k_{max}) = 0$$

where:

$$k_{max} = sup_k\{K\}$$

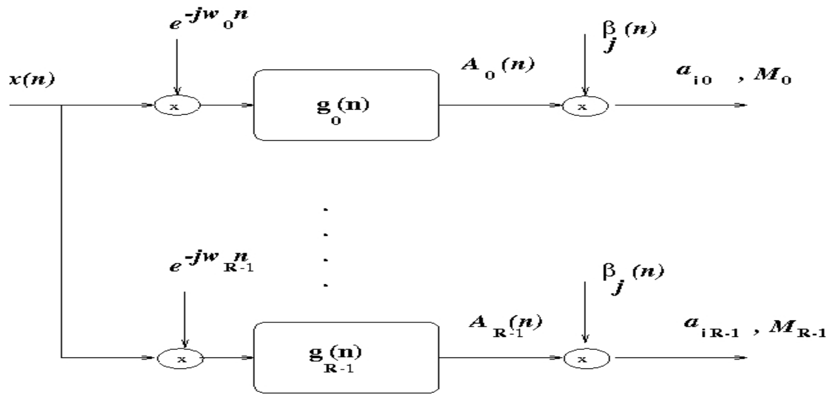


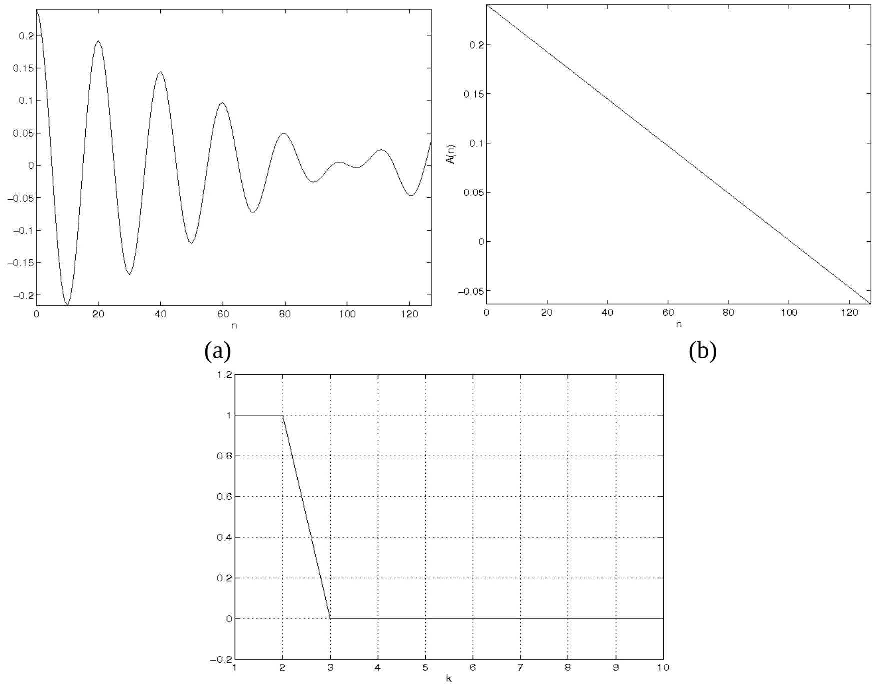

**Fig. 2. Multi-SV Parameters decomposition.**

Picking up the least upper bound of the estimated *M*'s, *i.e.*, $\hat{M} = max\{\hat{M}_0, \hat{M}_1, ..., \hat{M}_{R-1}\}$, to be the order of expansion of the EP estimator over the whole TF domain of the ES which implies that $a_{rk}$=0 for $k>M_r$. Another way of estimating the ES is to use different values of *M* for different ranges of frequencies according to the different values of the estimated *M* 's, *i.e.*, dividing the ES into a couple of bands each with different values of expansion order *M*. Two difficulties with this algorithm are: 1): the need to know the frequency of the sinusoidal components of *x(n)*, 2): since in practice we would not know the original polynomials, the value of *M* depends on the polynomials used to represent *A(n)*. The first problem can be remedied if the phase of each component is linear. When this is the case, the frequency of each sinusoidal can be estimated from the frequency spectrum of the process. For the second problem, one would expect different values for different sets of polynomials. The above algorithm is illustrated by means of some examples in the following section.

## 3. Simulations

### *3.1 Single SV Parameter Examples*

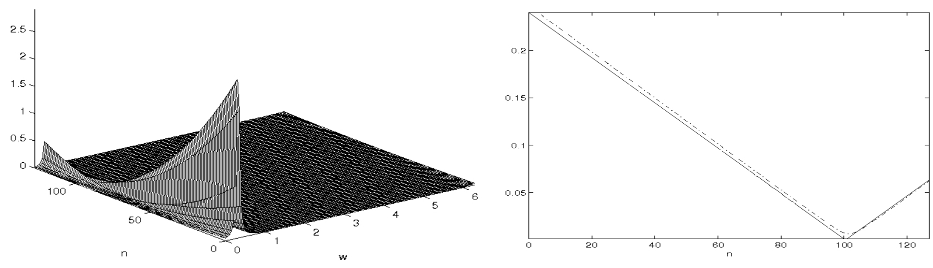
Sequences such as (14) may represent amplitude-modulated harmonics such as those arising in radar, sonar, and communication systems; an example occurring in active sonar systems is found in Ref. [8]. Assuming we have the non-stationary process defined as in Eq. (14) with $w_0$=0.1π *radians* and the SV parameter *A(n)* is expanded using only two OPs $\{b_i(n)\}_{i=1..M_0}$ such that *A(n)* is defined as in Eq. (15) with $\{a_i = 1\}_{i=1..M_0}$ and $M_0$=2. Figure 3(a) depicts the real part of *x(n)* and Fig. 3(b) shows the time variation of *A(n)*. From Eq. (17) we get the estimated decomposition coefficients $\{a_i\}$ as depicted in Fig. 3(c).




(a) (b) (c)

**Fig. 3: (a) The real part of *x(n)*, (b) *A(n)* for *M=2* and $\{a_i=1\}_{i=1,2}$, (c) The estimated expansion coefficients of *A(n)* when *M=2*.**

From the above figure we observe that the coefficients of the decomposition go down to zero when the index *k* exceeds the minimum value of *M*, i.e. when it exceeds *k=2*. This concludes that to track *A(n)*, we use an expansion order *M=2*. The ES of *x(n)* is shown in Fig. 4(a) when the expansion order *M*=2 is used. Fig. 4(b) shows the absolute value of the estimated amplitude compared with the absolute value of the actual one at the frequency of interest.




(a) (b)

**Fig. 4: (a) The ES of *x(n)* with *M* =2, (b) $\hat{A}(n)$ (dotted) vs. *A(n)* (solid) when *M* =2.**

Consider now the above example but the expansion order of the SV parameter *A(n)* is increased such that it can be expanded using six OPs $\{b_i(n)\}_{i=1..6}$. In this case, assume that $w_0 = 0.7p$ and $\{a_i=1\}_{i=1..\,6}$. Fig. 5(a) depicts the real part of *x(n)* while Fig. 5(b) shows the time variation of *A(n)*. Using a band-pass filter (BPF) centered at *w*=0.7π, then after demodulating its output by $e^{-j0.7pn}$ we get the SV parameter *A(n)* and then decompose it to get the expansion coefficients shown in Fig. 5(c). From Fig. 5(c) it is clear that the coefficients of the decomposition go down to zero when the minimum value of *M* is exceeded, *i.e.*, when *k*>6. This concludes that to be able to track the SV parameter *A(n)* of the non-stationary process *x(n)*, the expansion order of the ES should be *M=6*.

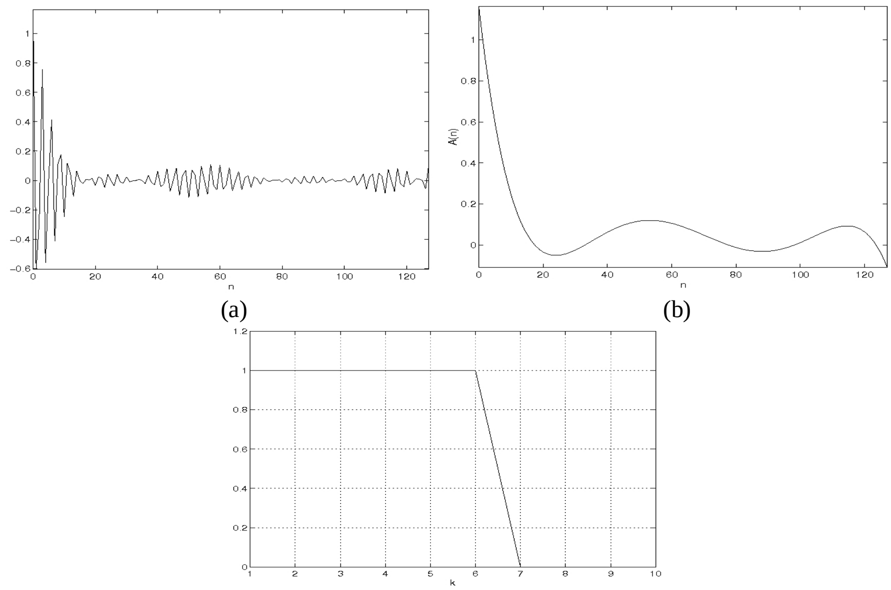



**Fig. 5: (a) The real part of *x(n)*, (b) *A(n)* for *M* =6 and $\{a_i=1\}_{i=1..\,6}$ ,(c) The estimated expansion coefficients $\{a_k\}$ for *M*=6 and $\{a_i=1\}_{i=1..\,6}$.**

The ES of *x(n)* is depicted in Fig. 6(a) with *M*=6. Figures 6(b), 6(c), 7(a), and 7(b) show the absolute value of *A(n)* compared to the absolute value of the estimated ones, with order of expansions *M*=2,4,6, and 8, respectively. It is clear from the figures that when the expansion order *M* increases, $\hat{A}(n)$ becomes closer to *A(n)*, then they coincide when it exceeds 6. This is obvious from Fig. 7(b) where *M*=8.

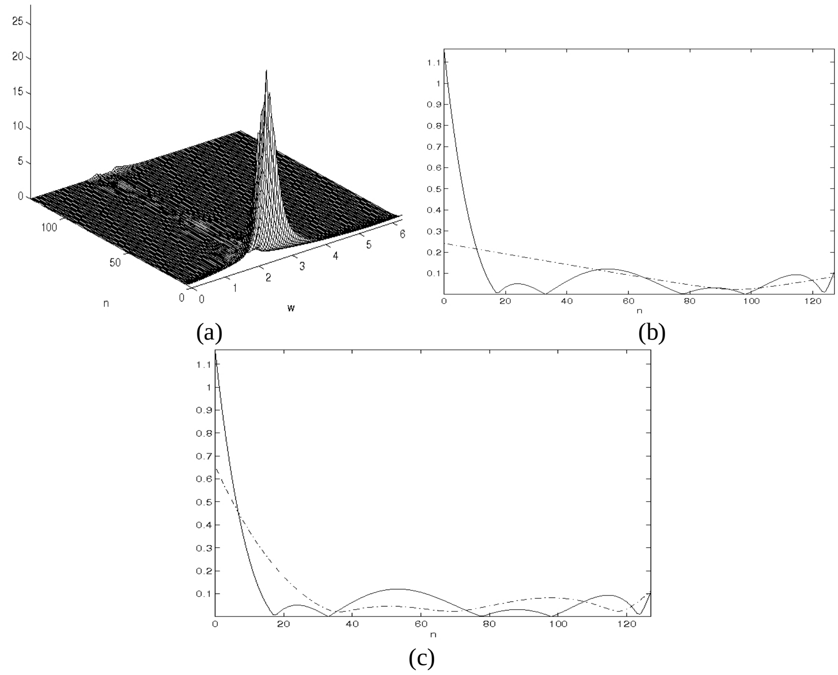


**Fig. 6: (a) The ES of *x(n)* with *M*=6, (b) $\hat{A}(n)$ (dotted-line) vs. *A(n)* (solid-line) when *M*=2, (c) $\hat{A}(n)$ (dotted-line) vs. *A(n)* (solid-line) when *M*=4.**

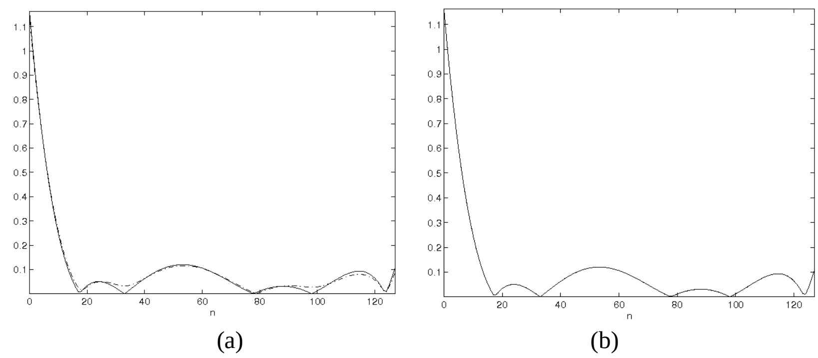


**Fig. 7: (a) $\hat{A}(n)$ (dotted-line) vs. *A(n)* (solid-line) when *M*=6, (b) $\hat{A}(n)$ (dotted-line) vs. *A(n)* (solid-line) when *M*>6.**

### *3.2 Multi-SV Parameters Examples*

To illustrate the proposed approach, let us define a non-stationary process *x(n)* such that

$$x(n) = \sum_{k=0}^{R-1} A_k(n) e^{j w_k n} \tag{24}$$

where: $R$ is the number of sinusoidals in the process and is equal to two, $w_0$ is the angular frequency of the first sinusoid and is equal to $0.1\pi$, $w_1$ is the angular frequency of the second sinusoid and is equal to $0.7\pi$, and the SV parameter of each sinusoid is represented as

$$A_k(n) = \sum_{i=1}^{M_k} a_{ki} b_i(n) \quad 0 \le k \le R-1 \tag{25}$$

with $M_0$=2, $M_1$=6, and $\{a_{ik} = 1\}_{\forall i,k}$ . The real part of $x(n)$ is depicted in Fig. 8(a).

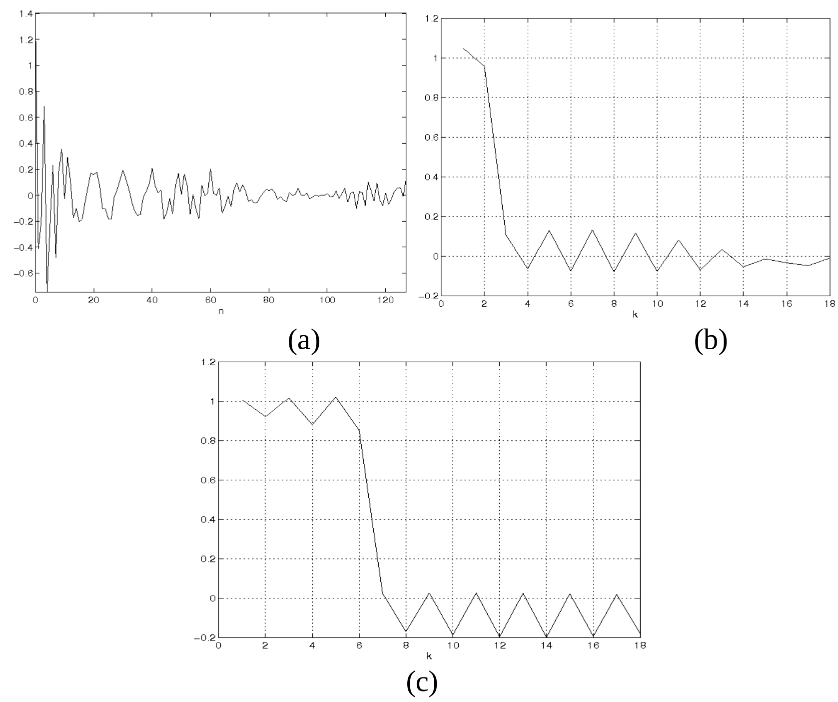



**Fig. 8: (a) The real part of $x(n)$ when $M_0$=2, $M_1$=6, and $\{a_{ik} = 1\}_{\forall i,k}$ (b) The estimated expansion coefficients, $\hat{a}_{ik}$, for $\{a_{ik}=1\}$ when $M_0$=2, (c) The estimated expansion coefficients, $\hat{a}_{ik}$, for $\{a_{ik} = 1\}_{\forall i,k}$ when $M_1$=6.**

Demodulating $x(n)$ by $w_0$, and then by $w_1$ we get $x_0(n)$ and $x_1(n)$, respectively, then, as shown in Fig. 9, passing these processes through the low pass filters $g_0(n)$ and $g_1(n)$, respectively.

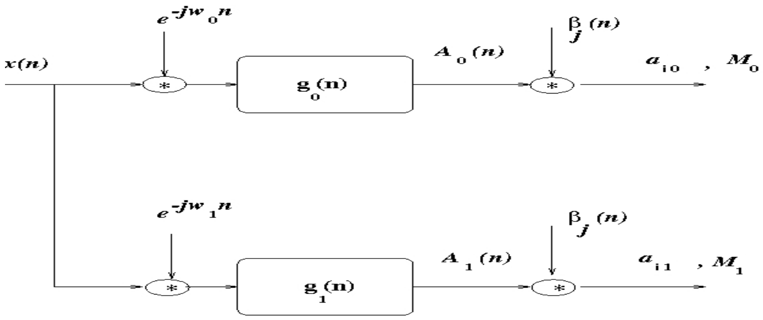
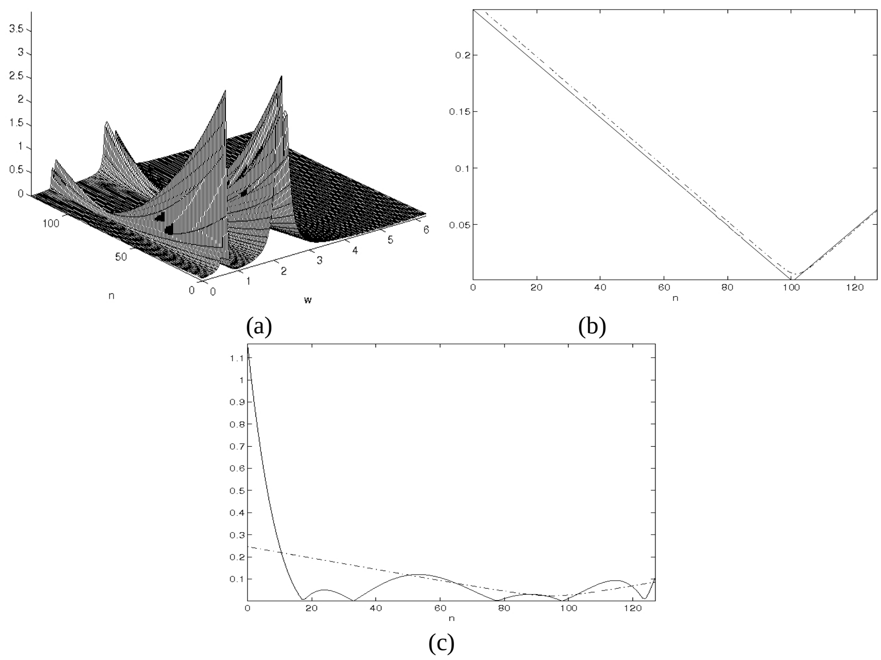



**Fig. 9. An example of decomposing a signal with two SV parameters.**

The filtering process is carried out under the condition that the bandwidths $BW_{A_0}(n) \le BW_{g_0}(n)$ and $BW_{A_1}(n) \le BW_{g_1}(n)$. Using an LPF with a cut-off frequency $w_c=0.15\pi$, the outputs of the filters are $A_0(n)$ and $A_1(n)$, respectively.



(a) (b)

(c)

**Fig. 10: (a) The ES of *x(n)* using *M*=2 as an expansion order, (b) $\hat{A}_0(n)$ (dotted-line) vs. $A_0(n)$ (solid-line) when *M*=2, (c) $\hat{A}_1(n)$ (dotted-line) vs. $A_1(n)$ (solid-line) when *M*=2.**

Decomposing each one of $A_0(n)$ and $A_1(n)$ separately, from Fig. 8(b) and 8(c), we get the estimated $\hat{M}_0$ *and* $\hat{M}_1$ to be approximately 2 and 6, respectively. Therefore, $\hat{M}$ =6 is the minimum expansion order of the EP estimator of *x(n)*. We observe from Figs. 8(b) and 8(c) the effects of the filtering process on the estimates of the *M*'s as some ripples through the variation of *k*. Fig. 10 (a) shows the ES of *x(n)* when the expansion order *M*=2 is used. Figures 10(b) and 10(c) show the absolute values of $A_0(n)$ and $A_1(n)$ compared with $\hat{A}_0(n)$ and $\hat{A}_1(n)$, respectively, when *M*=2 is used as the expansion order. For these two figures, we observe that $\hat{A}_0(n)$ is close to $A_0(n)$. However $\hat{A}_1(n)$ is not comparable to $A_1(n)$. This is due to the fact that the expansion order *M*=2 is not enough for representing $A_1(n)$ since it needs an expansion order *M* >6 to be able to track it. Therefore, the expansion order must be increased to track the variation of $A_1(n)$. The ES of *x(n)* is depicted in Fig. 11(a) using an expansion order *M*=6. Fig. 11(b) and 11(c) show the absolute values of $A_0(n)$ and $A_1(n)$ compared with $\hat{A}_0(n)$ and $\hat{A}_1(n)$, respectively, when *M*=6 is used as the expansion order. It is clear in this case that an expansion order *M*=6 is necessary to reasonably represent $A_1(n)$.

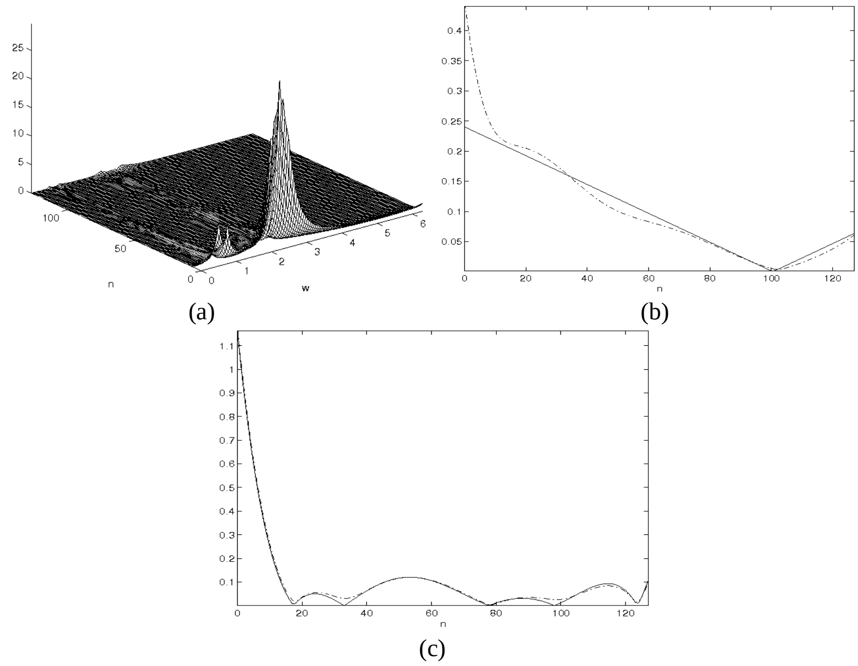



**Fig. 11: (a) The ES of *x(n)* using *M*=6 as an expansion order, (b) $\hat{A}_0(n)$ (dotted-line) vs. $A_0(n)$ (solid-line) when *M*=6, (c) $\hat{A}_1(n)$ (dotted-line) vs. $A_1(n)$ (solid-line) when *M*=6.**

### *3.2 EB and Nonstationary Noise*

To show the power of the EB in removing the non-stationary additive Gaussian noise, consider this example. According to Fig. 1, we have

$$z(n) = y(n) + h(n)$$

Let *y(n)* be represented as

$$y(n) = A(n)e^{jw_0 n}$$

where *A(n)* is decomposed as

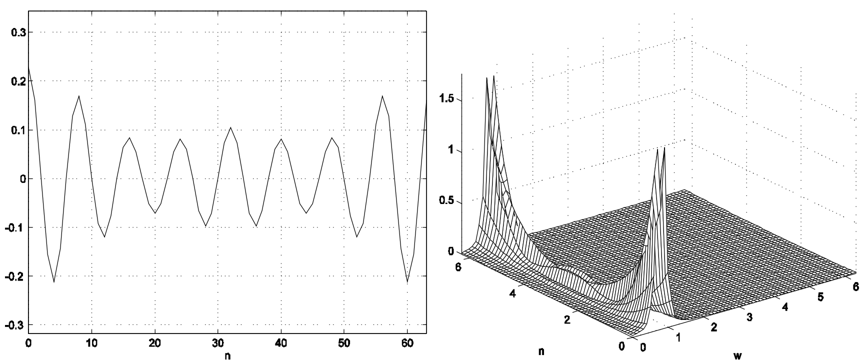
$$A(n) = \sum_{i=0}^{M-1} a_i b_i(n)$$

with $M = 3, a_i = 1/i$, $w_0 = .25p$, and $\{b_i(n)\}$ is a set of orthonormal polynomials (the Fourier orthonormal polynomials are considered). Also let $h(n)$ be a zero-mean, non-stationary Gaussian noise generated as follows:

$$h(n) = d(n)w(n)$$

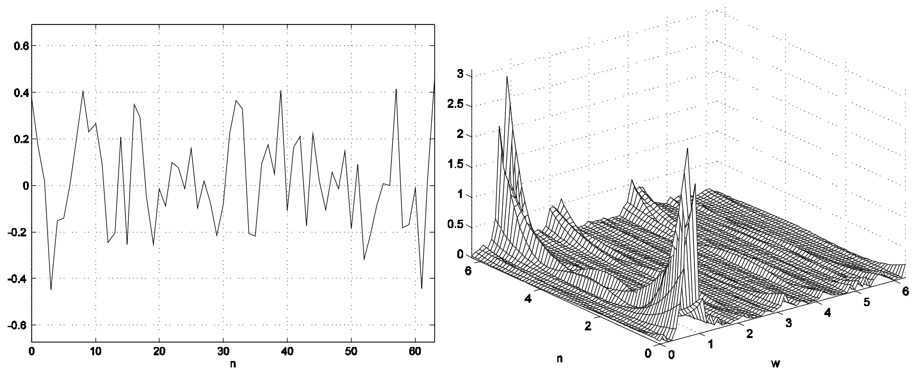
where *d(n)* is a deterministic signal and *w(n)* is a zero-mean, unit variance stationary Gaussian noise and independent of *y(n)*. Obviously, $E\{h(n)\}=0$ and $s_h^2(n) = d^2(n)s_w^2 = d^2(n)$.

The real part of the non-stationary signal *y(n)* is shown in Fig. 12(a) and its ES is shown in Fig. 12(b). The real part of the noisy signal z*(n)* is shown in Fig. 13(a). Figure 13(b) displays the ES of the noisy signal with SNR= 0 dB. Using 60 Monte-Carlo runs in such a way that $z^{(i)}(n) = y(n) + h^{(i)}(n), i = 1,2,...,60$, Fig. 14(a) shows the real part of the reconstructed signal $\hat{y}(n)$ versus the original one, *y(n)*, and Fig. 14(b) displays the ES of $\hat{y}(n)$ after utilizing the evolutionary bispectrum to remove the noise.



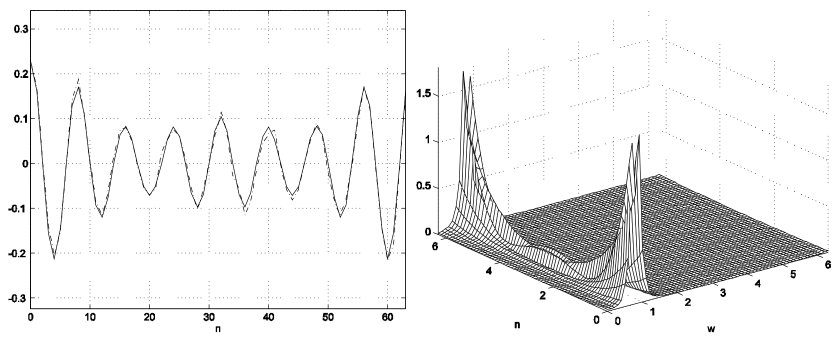

(a) (b)

**Fig. 12: (a) The real part of *y(n)*, (b) The ES of *y(n)*.**




(a) (b)

**Fig. 13: (a) The real part of *z(n)* with SNR= 0 dB, (b) The ES of *z(n)*.**




(a) (b)

**Fig. 14: (a) The real part of $\hat{y}(n)$ vs. *y(n)* (smooth), (b) The ES of the signal $\hat{y}(n)$ processed by the EB.**

## 4. Conclusions

In this paper, an algorithm for estimating the SV parameters, and hence, the TF kernel of a non-stationary process by using orthogonal projection theory was proposed. This estimated TF kernel could be used for estimating the ES of a non-stationary process with an order based on the dimensionality of the expanding OPs. In the proposed method the process was decomposed to basic elements by projecting it, recursively, onto elementary OPs. Two cases were considered: Single and multiple SV parameters. Some simulations were presented when the process had either single or multiple SV parameters. Moreover, since the evolutionary bispectrum of a process is identically zero, it could be used to remove such noise when it is symmetrically distributed.

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# تقدير الطيف التطوري والطيف الثنائي لإشارة غير مستقرة

**عبدالله إبراهيم الشوشان**

*قسم علوم الحاسب - كلية الحاسب والمعلومات - جامعة القصيم -*
*المملكة العربية السعودية*
*shoshan@myway.com*

*المستخلص*. في هذا البحث، تم اقتراح طريقة لتخمين العوامل المتغيرة المخفية داخل الإشارات غير المستقرة، والتي تتغير مع الزمن عن طريق الاسقاطات المتعامدة. ونستفيد من هذه الطريقة في مجموعة مهمة من التطبيقات، من أهمها التطبيقات الخاصة بتخمين دوال الزمن-التردد أو ما يمكن تسميته الطيف التطوري للإشارة، والتي يتم تطبيقها لمعرفة خصائص الإشارات غير المستقرة مع الزمن، حيث يتم في هذه الطريقة إيجاد درجة إحداثيات الإشارة، ومن ثم درجة استقرارها. ويتم معرفة درجة عدم استقرار الإشارة عن طريق الإسقاط المتكرر باستخدام مجموعة من الدوال المتعامدة، حيث إن هذه الدوال تمثل أساس عناصر الإشارة، وزيادة عدد هذه الدوال يوضح زيادة درجة عدم الاستقرار. وفي طيات هذا البحث، تم تطبيق الطريقة المقترحة في عدة أمثلة بيانية توضيحية، أثبتت كفاءة الطريقة. إضافة إلى ذلك فقد تم اقتراح استخدام الطيف الثنائي(Evolutionary Bispectrum) لإزالة تأثير التشويشات ذات التوزيع التماثلي، سواء كانت خصائص الإشارة مستقرة أو متغيرة مع الزمن.